INFINITE CW-COMPLEXES, BRAUER GROUPS AND PHANTOM COHOMOLOGY

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2 May 2013

Abstract. Expanding a result of Serre on finite CW-complexes, we show that the Brauer group coincides with the cohomological Brauer group for arbitrary compact spaces. Using results from the homotopy theory of classifying spaces for Lie groups, we give another proof of the result of Antieau and Williams that equality does not hold for Eilenberg–MacLane spaces of type $K(\mathbb{Z}/n\mathbb{Z}, 2)$. Employing a result of Dwyer and Zabrodsky, we show the same for the classifying spaces BG where G is an infinite-dimensional $\mathbb{F}_p$-vector space. In this context, we also give a formula expressing phantom cohomology in terms of homology.

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Introduction

Generalizing and unifying classical constructions, Grothendieck [23] introduced the *Brauer group* $\mathrm{Br}(X)$ and the *cohomological Brauer group* $\mathrm{Br}'(X)$ for arbitrary ringed spaces and ringed topoi. Roughly speaking, the elements in the former are equivalence classes of geometric objects, which can be regarded, among other things, as PGL_n-bundles. In contrast, elements in the latter are cohomology classes in degree two of finite order, with coefficients in the multiplicative sheaf $\mathcal{O}_X^\times$ of units in the structure sheaf. The machinery of nonabelian cohomology yields an inclusion

$$\mathrm{Br}(X) \subset \mathrm{Br}'(X),$$

and Grothendieck raised the question under which circumstances this inclusion is an equality. This question is particularly challenging in algebraic geometry, where one works with the étale site of a scheme X. According to an unpublished result of Gabber, for which de Jong [12] gave an independent proof, equality $\mathrm{Br}(X) =$

2000 *Mathematics Subject Classification.* 55N10, 16K50, 14F22.

$\mathrm{Br}'(X)$ holds for quasiprojective schemes. For many schemes, for example smooth threefolds without ample sheaves, the question is regarded as wide open. Note that there are "trivial" counterexamples bases on nonseparated schemes ([14], Corollary 3.11) and that equality holds for normal algebraic surfaces [44] and complex smooth surfaces [45].

The goal of this paper is to study Grothendieck's question in a purely topological settings, where X is a CW-complex, endowed with the sheaf of continuous complex-valued functions. In this situation, the cohomological Brauer group $\mathrm{Br}'(X)$ can be identified with the torsion part of

$$\mathrm{Ext}^1(H_2(X)/\mathrm{Divisible}, \mathbb{Z}).$$

According to a result of Serre outlined in [23], equality $\mathrm{Br}(X) = \mathrm{Br}'(X)$ holds for *finite CW-complexes*. Slightly expanding Serre's result, we show:

Theorem. *For each compact space X, we have $\mathrm{Br}(X) = \mathrm{Br}'(X)$.*

A construction of Bödigheimer [8] involving "long spheres" then implies that for every torsion group T, there is indeed a compact space—usually not admitting a CW-structure—whose Brauer group is T.

The main part of this paper, however, is concerned with infinite CW-complexes. Then equality between Brauer group and cohomological Brauer group does not necessarily hold:

Theorem (Antieau and Williams)**.** *Let X be an Eilenberg–MacLane space of type $K(\mathbb{Z}/n\mathbb{Z}, 2)$. Then the the Brauer group $\mathrm{Br}(X)$ vanishes, whereas the cohomological Brauer group $\mathrm{Br}'(X)$ is cyclic of order n.*

Antieau and Williams [3] used multiplicative properties of the cohomology ring $H^*(\mathrm{PU}(n), \mathbb{Z})$ with respect to the torsion subgroup. The case $n = 2$ was already considered by Atiyah and Segal [6]. Unaware of these results, we had found another proof; after putting the first version of this paper onto the arXiv, Antieau and Williams informed us about [3].

Our approach is based on a fact from the homotopy theory of classifying spaces of connected simple Lie groups, stating that any self map $BG \to BG$ not homotopic to a constant map induces bijections on rational homology groups. Nontrivial homotopy classes of selfmaps indeed exists, namely the *unstable Adams operations* ψ^k first constructed by Sullivan [49], and further studied by Ishiguro [30], Notbohm [40], and Jackowski, McClure and Olivier [31].

We also show for arbitrary abelian groups G, any Brauer class on the Eilenberg–MacLane space of type $K(G, 2)$ must live in the torsion part of $\mathrm{Ext}^1(G/\mathrm{Torsion}, \mathbb{Z})$. This has a natural interpretation as *phantom classes*, that is, Brauer classes that become trivial on all finite subcomplexes. Our second main result is a purely algebraic description of such phantoms, a kind of Universal Coefficient Theorem, which might be useful in other contexts:

Theorem. *Let X be a CW-complex. Then there is a natural identification of phantom cohomology $H^n(X)_{\mathrm{ph}}$ with $\mathrm{Ext}^1(H_{n-1}(X)/\mathrm{Torsion}, \mathbb{Z})$.*

In turn, it is straightforward to construct infinite CW-complexes X of dimension three for which the phantom cohomological Brauer group are arbitrary divisible torsion groups. These also coincide with the Brauer group, since $\mathrm{Br}(X) = \mathrm{Br}'(X)$ for all CW-complexes of dimension at most four, by a result of Woodward [52].

It is easy to see that the cohomological Brauer group for Eilenberg–MacLane spaces of type $K(G,n)$, $n \geq 3$ vanishes. We finally analyze the case $n=1$, that is, classifying spaces $BG = K(G,1)$ for arbitrary discrete groups G. According to result of Kan and Thurston [33], every CW-complex is homotopy equivalent to the *plus construction* $(BG)^+$ for some group G, which is usually uncountable, with respect so some perfect normal subgroup $N \subset G$. In some sense, this reduces Grothendieck's question on the equality of $\mathrm{Br}(X) \subset \mathrm{Br}'(X)$ to the case $X = BG$. Our third main result is:

Theorem. *Let $X = BG$ be the classifying space of an infinite-dimensional $\mathbb{F}_p$-vector space G. Then* $\mathrm{Br}(X) \subsetneqq \mathrm{Br}'(X)$.

The proof relies on a result of Dwyer and Zabrodsky [13], who showed that every bundle over the classifying space of finite p-groups comes from a representation, together with some facts on unitary representations due to Backhouse and Bradley [7]. We actually proof a more general version, where G can be a p-primary torsion group whose basic subgroups $H \subset G$ are infinite.

The paper is organized as follows: In Section 1, we recall some well-known facts on Brauer groups in the topological context and show that $\mathrm{Br}(X) = \mathrm{Br}'(X)$ holds for compact spaces. Section 2 contains further preparatory material. In Section 3, contains a new proof that the Brauer group of an Eilenberg–MacLane space of type $K(\mathbb{Z}/n\mathbb{Z}, 2)$ vanishes. This naturally leads to Section 4, in which we discuss phantom cohomology and express it in terms of homology. We also realize arbitrary divisible torsion groups as phantom Brauer groups on 3-dimensional CW-complexes. In the final Section 5, we turn to classifying spaces $BG = K(G,1)$ for discrete groups. Here the main result is that for many infinite p-primary torsion groups, for example infinite-dimensional $\mathbb{F}_p$-vector spaces, the Brauer group of BG is strictly smaller than the cohomological Brauer group.

Acknowledgement. We thank Benjamin Antieau and Ben Williams for informing us about the papers [3], [4] and [52]. The second author wishes to thank Wilhelm Singhof for helpful discussions.

1. Topological Brauer groups

In this section we collect some useful facts on Brauer groups of topological spaces, many of which are well-known. Throughout, we shall encounter both sheaf and singular cohomology; if not indicated otherwise, cohomology groups are sheaf cohomology groups.

First suppose X is a general ringed topos. We write $\mathrm{PGL}_n(\mathcal{O}_X)$ for the sheaf of groups associated to the presheaf $U \mapsto \mathrm{PGL}_n(\Gamma(U, \mathcal{O}_X))$. As explained in [23], Section 1, the obstruction for extending a $\mathrm{PGL}_n(\mathcal{O}_X)$-torsor $\mathcal{T}$ to a $\mathrm{GL}_n(\mathcal{O}_X)$-torsor is a cohomology class $\alpha \in H^2(X, \mathcal{O}_X^\times)$. More precisely, one has an exact sequence

$$H^1(X, \mathcal{O}_X^\times) \longrightarrow H^1(X, \mathrm{GL}_n(\mathcal{O}_X)) \longrightarrow H^1(X, \mathrm{PGL}_n(\mathcal{O}_X)) \longrightarrow H^2(X, \mathcal{O}_X^\times) \tag{1}$$

of pointed sets coming from the theory of nonabelian cohomology (compare [21], Chapter V and [48], Chapter I, §5). We say that α is the *obstruction class* of the torsor $\mathcal{T}$; it might also be regarded as a *characteristic class*. Similarly, the obstruction against extending $\mathcal{T}$ to an $\mathrm{SL}_n(\mathcal{O}_X)$-torsor is a class $\tilde{\alpha} \in H^2(X, \mu_n(\mathcal{O}_X))$, which maps to α, revealing that $n \cdot \alpha = 0$. Following Grothendieck, we regard the

cohomological Brauer group $\mathrm{Br}'(X)$ as the torsion part of $H^2(X, \mathcal{O}_X^\times)$. In contrast, the *Brauer group*

$$\mathrm{Br}(X) \subset \mathrm{Br}'(X) \subset H^2(X, \mathcal{O}_X^\times)$$

is the subset of elements that are obstruction classes for some $\mathrm{PGL}_n(\mathcal{O}_X)$-torsors $\mathcal{T}$ for certain $n \geq 1$.

Another way to see this goes as follows: The $\alpha \in H^2(X, \mathcal{O}_X^\times)$ correspond to isomorphism classes of $\mathcal{O}_X^\times$*-gerbes* $\mathscr{X}$ over X, and α lies in the Brauer group if one may choose $\mathscr{X}$ as the gerbe of extensions to $\mathrm{GL}_n(\mathcal{O}_X)$-torsors for some $\mathrm{PGL}_n(\mathcal{O}_X)$-torsor $\mathcal{T}$ (see [18], Chapter V, §4). As explained in [12], this is equivalent to the existence of certain locally free *twisted sheaf* of rank n, where the twisting is with respect to some cocycle representing α.

Now let X be a topological space, and assume that the structure sheaf $\mathcal{O}_X$ is the sheaf of continuous complex-valued functions $\mathcal{C}_X$. The exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{C}_X \longrightarrow \mathcal{C}_X^\times \longrightarrow 1,$$

where the map on the left is $f \mapsto e^{2\pi i f}$, yields an exact sequence

$$H^2(X, \mathcal{C}_X) \longrightarrow H^2(X, \mathcal{C}_X^\times) \longrightarrow H^3(X, \mathbb{Z}) \longrightarrow H^3(X, \mathcal{C}_X).$$

Recall that a space X is *paracompact* if it is Hausdorff, and every open covering admits a refinement that is locally finite. For paracompact spaces, the sheaf $\mathcal{C}_X$ is soft by the Uryson Lemma, whence acyclic (see [19], Chapter II, Theorem 4.4.3), and we obtain:

Proposition 1.1. *For paracompact spaces X, there is a canonical identification of the cohomological Brauer group $\mathrm{Br}'(X)$ with the torsion part of $H^3(X, \mathbb{Z})$.*

Throughout, we are mainly interested in CW-complexes. Then there is a useful interpretation in terms of singular homology groups as well. For any abelian group G, let us write

$$\mathrm{Torsion}\, G \quad \text{and} \quad G/\mathrm{Divisible}$$

for the torsion subgroup, and the quotient by the maximal divisible subgroup, respectively.

Proposition 1.2. *If X is a CW-complex, then there is a canonical identification*

$$\mathrm{Br}'(X) = \mathrm{Torsion}\, \mathrm{Ext}^1(H_2(X), \mathbb{Z}),$$

and this equals the torsion part of $\mathrm{Ext}^1(H_2(X)/\mathrm{Divisible}, \mathbb{Z})$ *as well.*

Proof. CW-complexes are paracompact by Miyazaki's result, see [16], Theorem 1.3.5. Moreover, sheaf cohomology for the sheaf of locally constant integer-valued functions coincides with singular cohomology, compare [9], Chapter III, Section 1. The Universal Coefficient Theorem gives a short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(H_2(X), \mathbb{Z}) \longrightarrow H^3(X) \longrightarrow \mathrm{Hom}(H_3(X), \mathbb{Z}) \longrightarrow 0.$$

The term on the right is torsionfree, whence the the map on the left is bijective on torsion parts.

Finally, let $D \subset H_2(X)$ be the maximal divisible subgroup. Since divisible groups are injective objects, this is a direct summand, and it remains to check that $\mathrm{Ext}^1(D, \mathbb{Z})$ is torsion free. Now any divisible group is a direct sum of groups of the form $\mathbb{Q}$ and $\mathbb{Z}[p^{-1}]/\mathbb{Z}$. But $\mathrm{Ext}^1(\mathbb{Q}, \mathbb{Z})$ is a $\mathbb{Q}$-vector space (in fact, of dimension

$2^{\aleph_0}$, compare [53]), whence torsion free, and $\operatorname{Ext}^1(\mathbb{Z}[p^{-1}]/\mathbb{Z},\mathbb{Z})$ is isomorphic to the group of p-adic integers $\mathbb{Z}_p$ (see [50], p. 74), which is torsion free as well. $\square$

Let us record the following immediate consequence:

Corollary 1.3. *Let X be a CW-complex such that $H_2(X)$ is a direct sum of a divisible group and a free abelian group. Then* $\operatorname{Br}'(X) = 0$.

For an algebraic characterization of abelian groups G with torsion free $\operatorname{Ext}^1(G,\mathbb{Z})$, we refer to [15], Theorem 2.13. Under suitable suitable set theoretical assumptions, this are precisely the sums of divisible and free groups [37].

Concerning the Brauer group, one can say the following. First observe that the sheaf of groups $\mathrm{PGL}_n(\mathcal{C}_X)$ may be regarded as the sheaf of continuous PGL_n-valued functions. By abuse of notation, we write PGL_n for the Lie group $\mathrm{PGL}_n(\mathbb{C})$ of $n\times n$ invertible matrices modulo scalar matrices. There is a well-known equivalence of categories between the category of locally trivial principal PGL_n-bundles $P\to X$ and the category of $\mathrm{PGL}_n(\mathcal{C}_X)$-torsors $\mathcal{T}$, sending P to the sheaf of sections $\mathcal{T}$ (compare [21], Proposition 5.1.1).

In practice, it is sometimes convenient to work with more geometric objects instead. Let $f : V \to X$ be a continuous map whose fibers $f^{-1}(x)$, $x \in X$ are homeomorphic to $\mathbb{CP}^{n-1}$. A *chart* (U,ψ) consists of an open subset $U \subset X$ and an U-homeomorphism $\psi : f^{-1}(U) \to \mathbb{CP}^{n-1}\times U$. Two charts (U,ψ) and (U',ψ') are *compatible* if the transition maps $\psi'\circ\psi^{-1}$ take values in PGL_n fiberwise over $U\cap U'$. An *atlas* is a collection of compatible charts (U_i,ψ_i) with $X=\bigcup_i U_i$. A $\mathbb{CP}^{n-1}$-*bundle* in a continuous map $f : V\to X$ as above, endowed with a maximal atlas. One may also view them also as *relative Brauer–Severi-varieties* over X in the sense of Hakim [24]. We have an equivalence of categories between the category of locally principal PGL_n-bundles $P \to X$ and the category of $\mathbb{CP}^{n-1}$-bundles $V \to X$, sending the locally trivial principal bundle P to the associated locally trivial fiber bundle V defined as the quotient of $\mathbb{CP}^{n-1}\times P$ by the diagonal left action of PGL_n (compare [29], Chapter 4).

Throughout, we shall write $H^1(X,\mathrm{PGL}_n)$ for the set of isomorphism classes of objects $\mathcal{T}$, P, V in their respective categories. We simply use the term PGL_n-*bundle*, or just *projective bundle* when it is clear from the context to which objects we refer. The operation $\mathrm{PGL}_m\times\mathrm{PGL}_n\to\mathrm{PGL}_{mn}$ that comes from tensor product of matrices induces a pairing

$$H^1(X,\mathrm{PGL}_m)\times H^1(X,\mathrm{PGL}_n)\to H^1(X,\mathrm{PGL}_{mn}),$$

which in turn gives the addition in the Brauer group. We shall write $P\otimes P'$ and $V\otimes V'$ for the corresponding operations on bundles, respectively.

Stabilizing with respect to tensor products with trivial bundles leads to the colimit $\mathrm{PGL}=\mathrm{PGL}_\infty$. Note that this stabilization is different from the stabilization for vector resp. GL_n-bundles denoted by $\mathrm{GL}=\mathrm{GL}_\infty$ and given by adding trivial bundles. These different stabilizations are the reason why $\pi_{2k}(\mathrm{BPGL})\cong\mathbb{Q}$ is different from $\pi_{2k}(\mathrm{BGL})\cong\pi_{2k}(\mathrm{BU})\cong\mathbb{Z}$ for all $k\geq 1$ even if $\pi_{2k}(\mathrm{BPGL}_n)\cong \pi_{2k}(\mathrm{BGL}_n)\cong\pi_{2k}(\mathrm{BU}(n))$. Finally, note that tensoring a bundle with a trivial bundle does not change its obstruction class.

Given a topological group G admitting a CW-structure, we denote by BG the classifying space for numerable principal G-bundles. We regard it as a connected CW-complex homotopy equivalent to the Milnor's join construction $G\star G\star\ldots$,

together with a universal bundle, and write $[X, BG]$ for the pointed set of homotopy classes of continuous maps.

We are mainly interested in the case $G = \mathrm{PGL}_n$ and $G = \mathrm{PU}(n)$. The latter is the quotient of the unitary group $U(n)$ by the diagonally embedded circle group $U(1) = S^1$, which is also the quotient of the special unitary group $\mathrm{SU}(n)$ by the diagonally embedded group of n-th roots of unity $\mu_n = \mu_n(\mathbb{C})$.

Proposition 1.4. *Let X be a paracompact space. Then we have*

$$H^1(X, \mathrm{PGL}_n) = [X, \mathrm{BPGL}_n] = [X, \mathrm{BPU}(n)].$$

In particular, the canonical map $\mathrm{BPU}(n) \to \mathrm{BPGL}_n$ is a homotopy equivalence.

Proof. Since X is paracompact, every locally trivial principal PGL_n-bundle is numerable, and the first equality follows. The obvious inclusion $\mathrm{PU}(n) \subset \mathrm{PGL}_n$ is a deformation retract by Gram-Schmidt, hence a homotopy equivalence. As $\Omega BG \simeq G$ for any Lie group, it follows that $\mathrm{BPU}(n) \to \mathrm{BPGL}_n$ is a weak equivalence, thus a homotopy equivalence. $\square$

Remark 1.5. Some non-numerable locally trivial principal bundles over non-paracompact Hausdorff spaces are described in [10] and [46].

Let us record the following fact, which is essentially Serre's result on the Brauer group of finite CW-complexes:

Theorem 1.6. *Let X be a compact space. Then $\mathrm{Br}(X) = \mathrm{Br}'(X)$.*

Proof. Let $\alpha \in \mathrm{Br}'(X) \subset H^3(X, \mathbb{Z})$ be a cohomology class, say of order $n \geq 1$. Chose a lift $\tilde{\alpha} \in H^2(X, \mathbb{Z}/n\mathbb{Z})$. On paracompact spaces, sheaf cohomology coincides with Čech cohomology, according to [19], Chapter II, Theorem 5.10.1. Moreover, Čech cohomology on paracompact spaces with countable coefficients can by described via homotopy classes of continuous maps into Eilenberg–MacLane spaces, according to Huber's result [28]. Let Y be an Eilenberg–MacLane space of type $K(\mathbb{Z}/n\mathbb{Z}, 2)$, and $f : X \to Y$ a continuous map representing $\tilde{\alpha}$. The image $f(X) \subset Y$ is compact, whence contained in a finite subcomplex $Y' \subset Y$. By Serre's result [23], Theorem 1.6, the restriction of the universal cohomology class to Y' is the obstruction of some projective bundle $V' \to Y'$, and the pullback $f^*(V')$ shows that α lies in the Brauer group. $\square$

From this we get an amusing existence result:

Corollary 1.7. *For every abelian torsion group T, there is a compact space X so that $\mathrm{Br}(X)$ is isomorphic to T and that the inclusion $\mathrm{Br}(X) \subset \mathrm{Br}'(X) \subset H^3(X, \mathbb{Z})$ are equalities.*

Proof. Using "long spheres", Bödigheimer [8] constructed compact spaces realizing preassigned Čech cohomology groups. In particular, there is a compact space X with $\check{H}^3(X, \mathbb{Z}) \simeq T$. But on paracompact spaces, Čech cohomology coincides with sheaf cohomology, by [19], Chapter II, Theorem 5.10.1, and the result follows. $\square$

Note that for infinite abelian torsion groups T, there can be no CW structure on X. For otherwise there are only finitely many cells by compactness, and $H^3(X)$ would be finitely generated by the cellular cochain complex.

2. Some infinite CW-complexes

We start by constructing a concrete projective bundle on a particular CW complex Y of dimension three. Fix some $n \geq 1$, and let Y be the space obtained by attaching a 3-cell to the 2-sphere along a continuous map $\varphi : S^2 \to S^2$ of degree n. We thus have a cocartesian diagram

$$\begin{tikzcd} S^2 \arrow[r] \arrow[d, "\varphi"'] & D^3 \arrow[d, "\Phi"] \\ S^2 \arrow[r] & Y \end{tikzcd}$$

and regard $Y = e^0 \cup e^2 \cup e^3$ as a CW-complex with three cells. The cellular cochain complex (see for example [51], Chapter II, Theorem 2.19), or the cellular chain complex together with the Universal Coefficient Theorem easily yields

$$\mathrm{Br}'(Y) = \mathrm{Torsion}\, H^3(Y) = H^3(Y) = \mathbb{Z}/n\mathbb{Z}.$$

By Serre's result, this group is generated by some $\mathrm{PGL}_{n'}$-bundle. The following argument shows that one may actually choose $n' = n$:

Let L be a complex line bundle on the 2-skeleton $Y^2 = S^2$ with Chern class $c_1(L) = 1$ (that is, the tautological bundle on $\mathbb{CP}^1$), where the Chern class is regarded as element of $H^2(S^2) = \mathbb{Z}$, and consider the complex vector bundle $E = L \oplus \mathbb{C}^{n-1}$ of rank n. Then

$$\varphi^*(L \oplus \mathbb{C}^{n-1}) = L^{\otimes n} \oplus \mathbb{C}^{n-1}.$$

Although the classifying map $Y^2 \to \mathrm{BGL}_n$ for E does not extend to Y, we have $L^{\otimes n} \oplus \mathbb{C}^{n-1} \simeq L \oplus \ldots \oplus L$. This is because vector bundles of rank n on the 2-sphere correspond, up to isomorphism, to elements in $\pi_1(\mathrm{GL}_n)$, the bundles in question have isomorphic determinant, and the determinant map $\mathrm{GL}_n \to \mathbb{C}^\times$ induces a bijection on fundamental groups. Since the projectivization of $L \oplus \ldots \oplus L$ is isomorphic to the product bundle $\mathbb{CP}^{n-1} \times S^2$, the classifying map $Y^2 \to \mathrm{BPGL}_n$ extends to Y, which yields a PGL_n-bundle $V \to Y$.

Proposition 2.1. *The obstruction class of the projective bundle $V \to Y$ constructed above generates the group $\mathrm{Br}'(Y)$.*

Proof. Fix $0 < i < n$. Since $\mathrm{Br}'(Y)$ is cyclic of order n, it suffices to check that $V^{\otimes i}$ is not the projectivization of a vector bundle. Suppose to the contrary that it is the projectivization of a vector bundle $E \to Y$. Obviously, it has rank in, and $\varphi^*(E)$ is trivial. In light of the exact sequence of pointed sets (1) applied to $X = Y^2$, we have $E|Y^2 \simeq (L \oplus \mathbb{C}^{n-1})^{\otimes i} \otimes N$ for some line bundle $N \to Y^2$. Consequently

$$\varphi^*(E) = (L^{\otimes n} \oplus \mathbb{C}^{n-1})^{\otimes i} \otimes N^{\otimes n}.$$

Taking first Chern classes, we obtain $0 = c_1\varphi^*(E) = in^i + in^{i+1}c_1(N)$, thus $n|i$, contradiction. ∎

Let us record a direct consequence for universal bundles:

Proposition 2.2. *Let $X = \mathrm{BPGL}_n$. Then the cohomological Brauer group $\mathrm{Br}'(X)$ is cyclic of order n, the obstruction class of the universal bundle $P \to X$ is a generator, and $\mathrm{Br}(X) = \mathrm{Br}'(X)$.*

Proof. Clearly, the Lie group PGL_n is connected. As $\mathrm{PSL}_n \to \mathrm{PGL}_n$ is the universal covering with fiber μ_n the nth roots of unity, its fundamental group is cyclic of order n. Thus the classifying space $X = \mathrm{BPGL}_n$ is simply connected, and Hurewicz shows that $H_2(X) = \pi_2(X) = \pi_1(\mathrm{PGL}_n) \simeq \mathbb{Z}/n\mathbb{Z}$. Thus the same holds for $\mathrm{Br}'(X)$ by Proposition 1.2. According to Proposition 2.1, there is a numerable PGL_n-bundle over some space whose obstruction class has order n. Thus the same holds for the universal bundle $P \to X$. If follows that $\mathrm{Br}(X) \subset \mathrm{Br}'(X)$ is an equality. □

According to a result of Woodward [52], see also [4], equality $\mathrm{Br}(X) = \mathrm{Br}'(X)$ holds for all CW-complexes of dimension ≤ 4; in fact, each cohomology class $\alpha \in \mathrm{Br}'(X)$ is the obstruction of some PGL_n-bundle $V \to X$ with $n = \mathrm{ord}(\alpha)$.

This applies, for example, to differential manifolds of dimension at most four, or the underlying topological space $X = S(\mathbb{C})$ of a algebraic $\mathbb{C}$-scheme or complex spaces S of complex dimension at most two. Note that all such spaces admit a CW-structure. It also applies to the underlying topological space $X = S(\mathbb{C})$ of complex Stein spaces S of complex dimension at most four, because such spaces have the homotopy type of a CW-complex of real dimension at most four by [25].

In higher dimensions, the following criterion reduces the problem to odd-dimensional CW-complexes:

Lemma 2.3. *Let X be a CW-complex of even dimension n, with $(n-1)$-skeleton $Y = X^{n-1}$. Let $\alpha \in \mathrm{Br}'(X)$ be a class whose restriction $\alpha|Y$ lies in the Brauer group $\mathrm{Br}(Y) \subset \mathrm{Br}'(Y)$. Then $\alpha \in \mathrm{Br}(X)$.*

Proof. The cases $n = 0, 2$ are trivial, for then $H_2(X)$ is free, so $\mathrm{Br}'(X)$ vanishes by Corollary 1.3. Now suppose $n \geq 4$. Choose some PGL_d-bundle $V \to Y$ whose obstruction class is $\alpha|Y$. Let $\varphi : \amalg S^{n-1}_\alpha \to Y$ be the attaching maps for the n-cells. The pullbacks $\varphi^*_\alpha(V)$ correspond to elements in

$$\pi_{n-2}(\mathrm{PGL}_d) = \pi_{n-2}(\mathrm{GL}_d) = \pi_{n-2}(U(d)).$$

Replacing V by the tensor product with a suitable trivial bundle, we may assume that $2d + 1 > n - 1$. The fibration $U(d+1)/U(d) = S^{2d+1}$ shows that we are in the stable range. We than have $\pi_{n-2}(U(d)) = \pi_{n-2}(U) = 0$, because n is even, compare [39], Section 3.1. Thus the pullbacks $\varphi^*_\alpha(V)$ are trivial, so the classifying map $Y \to \mathrm{BPGL}_d$ for V can be extended to X. The resulting bundle on X has α as obstruction class, since the restriction map $\mathrm{Br}'(X) \to \mathrm{Br}'(Y)$ is bijective. □

Proposition 2.4. *Let X be a finite-dimensional CW-complex that contains no cells of odd dimension $n \geq 5$. Then $\mathrm{Br}(X) = \mathrm{Br}'(X)$.*

Proof. Let $X^n \subset X$ be the n-skeleton. We check $\mathrm{Br}(X^n) = \mathrm{Br}'(X^n)$ by induction on $n \geq 4$. Equality holds for $n = 4$ by [52]. Now suppose $n \geq 5$, and that equality holds for $n-1$. If n is even, we may apply Lemma 2.3. If n is odd, then $X^n = X^{n-1}$, and equality holds as well. □

3. Eilenberg–MacLane spaces

We now examine the Brauer group of Eilenberg–MacLane spaces X of type $K(G, j)$, which we always regard as connected CW-complexes endowed with a universal cohomology class. For $j \geq 3$, such spaces have trivial cohomological Brauer group, according to Corollary 1.3. In particular, this holds for $K(\mathbb{Z}, 3)$, although for paracompact Y every torsion cohomology class in $H^3(Y, \mathbb{Z})$, which correspond to

an element of the cohomological Brauer group, arises as a pullback of the universal cohomology class on $K(\mathbb{Z},3)$.

The cases of interest are $j=1$ and $j=2$. By Proposition 1.2, an Eilenberg–MacLane space X of type $K(\mathbb{Q}/\mathbb{Z},2)$ has trivial cohomological Brauer group, although any torsion class in degree three on Y arises as the Bockstein of the pullback of the universal cohomology class. Similarly for the classifying space

$$\mathrm{BPGL}_\infty = K(\mathbb{Q}/\mathbb{Z},2)\times K(\mathbb{Q},4)\times K(\mathbb{Q},6)\times\ldots.$$

According to Proposition 1.2, the cohomological Brauer group of $K(\mathbb{Z}/n\mathbb{Z},2)$ is cyclic of order n. Antieau and Williams proved the following result, using multiplicative properties of the cohomology ring $H^*(PU(n),\mathbb{Z})$ with respect to the torsion subgroup ([3], Corollary 5.10):

Theorem 3.1. *The Brauer group of $K(\mathbb{Z}/n\mathbb{Z},2)$ is trivial.*

The special case $n=2$ was already considered by Atiyah and Segal, compare [6], proof of Proposition 2.1 (v). Another proof relying on the homotopy theory of classifying spaces that might be of independent interest appears at the end of this section.

What can be said about Eilenberg–MacLane spaces X of type $K(G,2)$, where the abelian group G is arbitrary? Then $H_2(X)=\pi_2(X)=G$, and Proposition 1.2 gives an inclusion $\mathrm{Br}'(X)\subset\mathrm{Ext}^1(G,\mathbb{Z})$.

Proposition 3.2. *Assumptions as above. With respect to the inclusion* $\mathrm{Br}'(X)\subset\mathrm{Ext}^1(G,\mathbb{Z})$*, the Brauer group is contained in the subgroup*

$$\mathrm{Ext}^1(G/\mathrm{Torsion},\mathbb{Z})\subset\mathrm{Ext}^1(G,\mathbb{Z}).$$

Proof. Let $T\subset G$ be the torsion subgroup. The short exact sequence of abelian groups $0\to T\to G\to G/T\to 0$ yields an exact sequence

$$\mathrm{Hom}(T,\mathbb{Z})\longrightarrow\mathrm{Ext}^1(G/T,\mathbb{Z})\longrightarrow\mathrm{Ext}^1(G,\mathbb{Z})\longrightarrow\mathrm{Ext}^1(T,\mathbb{Z})\longrightarrow 0$$

(recall that for abelian groups there are no higher Ext groups). The term on the left vanishes, hence the map on the right indeed is injective. Consider the collection $G_\alpha\subset G$ of all finite cyclic subgroups. This gives another short exact sequence $0\to H\to\bigoplus_\alpha G_\alpha\to T\to 0$ for some torsion group H. In turn, we get a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Ext}^1(G/T,\mathbb{Z}) \arrow[r] & \mathrm{Ext}^1(G,\mathbb{Z}) \arrow[r] \arrow[dr] & \mathrm{Ext}^1(T,\mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
 & & & \prod_\alpha \mathrm{Ext}^1(G_\alpha,\mathbb{Z}). &
\end{tikzcd}$$

Seeking a contradiction, we assume that there is a bundle $V\to X$ whose obstruction class viewed as an element $\alpha\in\mathrm{Ext}^1(G,\mathbb{Z})$ does not lie in the subgroup $\mathrm{Ext}^1(G/T,\mathbb{Z})$. By the preceding diagram, there must be an inclusion $\mathbb{Z}/n\mathbb{Z}\subset G$ so that the α restricts to a nonzero element in $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z},\mathbb{Z})$. Consider the induced map

$$f: K(\mathbb{Z}/n\mathbb{Z},2)\longrightarrow K(G,2)=X$$

of Eilenberg–MacLane spaces. Then $f^*(V)$ is a bundle with nontrivial obstruction class, in contradiction to Theorem 3.1. $\square$

We conjecture that the Brauer groups for arbitrary Eilenberg–MacLane spaces of type $K(G,2)$ vanish. In the next section, we shall see that elements from the group $\mathrm{Ext}^1(G/T,\mathbb{Z})$ correspond to so-called phantom classes.

Another proof for Theorem 3.1: Let X be an Eilenberg–MacLane space of type $K(\mathbb{Z}/n\mathbb{Z},2)$. Consider the Bockstein exact sequence

$$H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathbb{Z}/n\mathbb{Z}) \longrightarrow H^3(X,\mathbb{Z}) \xrightarrow{n} H^3(X,\mathbb{Z}).$$

The term on the left vanishes, by Hurewicz and the Universal Coefficient Theorem. In turn, the Bockstein map $H^2(X,\mathbb{Z}/n\mathbb{Z}) \to H^3(X,\mathbb{Z})$ is injective, and its image equals $\mathrm{Br}'(X) \subset H^3(X,\mathbb{Z})$, which is cyclic of order n. Denote by $\tilde{\alpha} \in H^2(X,\mathbb{Z}/n\mathbb{Z})$ the universal cohomology class, which generates this cyclic group, and $\alpha \in \mathrm{Br}'(X)$ the corresponding cohomological Brauer class. Seeking a contradiction, we suppose that there is a non-zero multiple $\beta = r\cdot\alpha$ lying in $\mathrm{Br}(X)$. Choose a PGL_m-bundle $V \to X$ whose obstruction class is β. As $m\cdot\beta = 0$, the number m is a multiple of $\mathrm{ord}(\beta) = n/\gcd(n,r)$. Tensoring V with the trivial $\mathrm{PGL}_{\gcd(n,r)}$-bundle, we may assume that $n|m$.

The existence of this bundle V has the following consequence: Let Y be an arbitrary paracompact space and $\gamma \in \mathrm{Br}'(Y)$ an element of order n. Viewing it as a torsion class in $H^3(Y,\mathbb{Z})$, we may choose some $\tilde{\gamma} \in H^2(Y,\mathbb{Z}/n\mathbb{Z})$ mapping to γ under the Bockstein. This $\tilde{\gamma}$ corresponds to a homotopy class of continuous maps $h : Y \to X$, and the PGL_m-bundle $h^*(V)$ on Y has obstruction $r\cdot\gamma$. We will reach a contradiction below by exhibiting a paracompact space Y and a class $\gamma \in \mathrm{Br}'(Y)$ of order n so that $r\cdot\gamma$ does not come from a PGL_m-bundle.

We construct such a space Y as a relative CW-complex by attaching a single cell to the classifying space $B = \mathrm{BPGL}_m$. Using that $\pi_i(B)$ are finite groups for $i \leq 3$, we infer with Serre's refined Hurewicz Theorem [47] that the canonical map $\pi_4(B) \to H_4(B)$ becomes bijective after tensoring with $\mathbb{Q}$. Since $\pi_4(B) \simeq \mathbb{Z}$, there is a continuous map $\varphi : S^4 \to B$ so that the induced map $H_4(S^4,\mathbb{Q}) \to H_4(B,\mathbb{Q})$ is bijective. The cocartesian diagram

$$\begin{tikzcd} S^4 \arrow[r] \arrow[d, "\varphi"'] & D^5 \arrow[d, "\Phi"] \\ B \arrow[r] & Y \end{tikzcd}$$

defines a relative CW-complex $B \subset Y$. Then we have a long exact sequence

$$H^i(Y,B) \longrightarrow H^i(Y) \longrightarrow H^i(B) \longrightarrow H^{i+1}(Y,B)$$

of relative singular cohomology groups with integral coefficients. The relative cohomology groups $H^i(Y,B)$ can be computed using the cellular cochain complex, thus vanish for $i \neq 5$ (see for example [51], Chapter II, Theorem 2.19). In particular, the restriction map $H^3(Y,\mathbb{Z}) \to H^3(B,\mathbb{Z})$ is bijective.

The obstruction class $\gamma_u \in \mathrm{Br}(B)$ of the universal bundle on B has order m, by Proposition 2.2. Let $\gamma \in \mathrm{Br}'(Y)$ be the unique class restricting to $m/n\cdot\gamma_u$, which has order n. As explained above, this yields a certain PGL_m-bundle $h^*(V)$ on Y with obstruction class $r\cdot\gamma$, which in turn is classified by a continuous map

$f : Y \to B = \mathrm{BPGL}_m$. Consider the commutative diagram

$$\begin{tikzcd}
S^4 \arrow[r] \arrow[d, "\varphi"'] & D^5 \arrow[d, "\Phi"] \arrow[dr] & \\
B \arrow[r] \arrow[rr, bend right, "g"'] & Y \arrow[r, "f"] & B.
\end{tikzcd}$$

The map $g \circ \varphi$ factors over D^5, whence is homotopic to a constant map. We infer that the map $g_* : H_4(B, \mathbb{Q}) \to H_4(B, \mathbb{Q})$ is zero.

On the other hand, $g : B \to B$ is not homotopic to a constant map, because $r \cdot \gamma | B = rm/n \cdot \gamma_u \neq 0$. Theorem 3.3 below implies that $g_* : H_4(B, \mathbb{Q}) \to H_4(B, \mathbb{Q})$ is bijective, contradiction. $\square$

We have just used the following result of Jackowski, Oliver and McClure [31]:

Theorem 3.3. *Let G be a compact, connected simple Lie group. Then for every nontrivial element $g \in [BG, BG]$, there is a positive integer k and an $\alpha \in \mathrm{Out}(G)$ such that $g = B\alpha \circ \psi^k$ where ψ^k is a so-called unstable Adams operation, having the property that $H^{2i}(\psi^k, \mathbb{Q})$ is multiplication by k^i.*

Actually, the units in the monoid of homotopy classes $[BG, BG]$ is the group of outer automorphisms $\mathrm{Out}(G)$, and the corresponding quotient has an embedding

$$[BG, BG]/[BG, BG]^\times \subset \mathbb{N}$$

into the multiplicative monoid of the integers. Its image consists of $k = 0$ and all $k > 0$ prime to the order of the Weyl group of G. These numbers $k > 0$ correspond to the unstable Adams operations ψ^k, which are defined in terms of the Galois action of $\mathrm{Gal}(\bar{\mathbb{Q}})$ on Grassmannians. First examples of unstable Adams operations were constructed by Sullivan [49]. This result apply in particular for the compact Lie group $G = \mathrm{PU}(n)$. Since $\mathrm{BPU}(n) \to \mathrm{BPGL}_n$ is a homotopy equivalence by Proposition 1.4 and $\mathrm{PU}(n)$ is a compact, connected simple Lie group, it applies to our situation.

4. Phantom cohomology

Let X be a Hausdorff space, and $X_\beta \subset X$, $\beta \in J$ the family of all compact subspaces, ordered by inclusion. In each degree $p \geq 0$, we define a subgroup $H^p(X)_{\mathrm{ph}} \subset H^p(X) = H^p(X, \mathbb{Z})$ by the exact sequence

$$0 \longrightarrow H^p(X)_{\mathrm{ph}} \longrightarrow H^p(X) \longrightarrow \varprojlim_{\beta \in J} H^p(X_\beta),$$

and call its elements *phantom cohomology classes*. By the very definition, a cohomology class is phantom if and only if it vanishes on each compact subspace. We are mainly interested in the case of CW-complexes X. Then one may replace the the system of all compact subspace by the cofinal system of all finite subcomplexes $X_\alpha \subset X$, $\alpha \in I$. We refer to McGibbon [36] and Rudyak [42], Chapter III for more on phantoms.

On CW-complexes, phantom cohomology classes may be described in a surprisingly simple way as follows:

Theorem 4.1. *Let X be a CW-complex. Then there is a natural identification*

$$H^n(X)_{\mathrm{ph}} = \mathrm{Ext}^1(H_{n-1}(X)/\mathrm{Torsion}, \mathbb{Z})$$

for each $n \geq 0$.

Proof. According to a generalization ([5], Corollary 12) of Milnor's exact sequence ([38], Lemma 2), we have a natural short exact sequence

$$0 \longrightarrow \varprojlim{}^1 H^{n-1}(X_\alpha) \longrightarrow H^n(X) \longrightarrow \varprojlim H^n(X_\alpha) \longrightarrow 0,$$

such that $H^n(X)_{\mathrm{ph}} = \varprojlim{}^1 H^{n-1}(X_\alpha)$. The Universal Coefficient Theorem gives short exact sequences

$$0 \longrightarrow \mathrm{Ext}^1(H_{n-2}(X_\alpha), \mathbb{Z}) \longrightarrow H^{n-1}(X_\alpha) \longrightarrow \mathrm{Hom}(H_{n-1}(X_\alpha), \mathbb{Z}) \longrightarrow 0.$$

The groups $H_{n-2}(X_\alpha)$ are finitely generated, since X_α are finite CW-complexes. Hence $\mathrm{Ext}^1(H_{n-2}(X_\alpha), \mathbb{Z})$ are non-canonically isomorphic to the torsion part of $H_{n-2}(X_\alpha)$, whence finite. According to [32], Corollary 7.2, the higher derived inverse limits of such groups vanish. Thus the map

$$\varprojlim{}^1 H^{n-1}(X_\alpha) \longrightarrow \varprojlim{}^1 \mathrm{Hom}(H_{n-1}(X_\alpha), \mathbb{Z})$$

in the long exact sequence for inverse limits is bijective. Clearly, we have

$$\mathrm{Hom}(H_{n-1}(X_\alpha), \mathbb{Z}) = \mathrm{Hom}(H_{n-1}(X_\alpha)/\mathrm{Torsion}, \mathbb{Z}).$$

Moreover, the canonical map $\varinjlim H_{n-1}(X_\alpha) \to H_{n-1}(X)$ is bijective. One easily sees that the induced map

$$\varinjlim (H_{n-1}(X_\alpha)/\mathrm{Torsion}) \to H_{n-1}(X)/\mathrm{Torsion}$$

is bijective as well. By Jensen's observation ([32], page 37), we have an identification

$$\varprojlim{}^1 \mathrm{Hom}(H_{n-1}(X_\alpha)/\mathrm{Torsion}, \mathbb{Z}) = \mathrm{Pext}^1(H_{n-1}(X)/\mathrm{Torsion}, \mathbb{Z}),$$

where the right hand side is the group of isomorphism classes of pure extensions. Recall that an extension in the category of abelian groups $0 \to G' \to G \to G'' \to 0$ is called *pure* if it remains exact after tensoring with arbitrary groups. The isomorphism classes of pure extensions form a subgroup

$$\mathrm{Pext}^1(G'', G') \subset \mathrm{Ext}^1(G'', G').$$

According to [17], §53.3, this subgroup is the first Ulm subgroup. Recall that for an abelian group E, the *first Ulm subgroup* is $U^1(E) = \bigcap_{m \geq 1}(mE)$. However, for any torsion free abelian group F, the group $\mathrm{Ext}^1(F, \mathbb{Z})$ is divisible (compare [17], page 223, point (I)), hence the latter coincides with its own first Ulm subgroup. Combining these observations, we infer

$$H^n(X)_{\mathrm{ph}} = \mathrm{Ext}^1(H_{n-1}(X)/\mathrm{Torsion}, \mathbb{Z})$$

as desired. □

This allows us to pinpoint how phantom classes must arise:

Corollary 4.2. *Let X be a CW-complex with either finitely many cells $e^{n-1} \subset X$ of dimension $n-1$, or finitely many cells of dimension $e^n \subset X$ of dimension n. Then $H^n(X)_{\mathrm{ph}} = 0$.*

Proof. Either of the two assumptions implies that the boundary map on the left in the cellular chain complex

$$H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}, X^{n-2}) \longrightarrow H_{n-2}(X^{n-2}, X^{n-3})$$

has finite rank. It easily follows that $H_{n-1}(X)$ is a direct sum of a finitely generated group and a free group. In turn $H_{n-1}(X)/\operatorname{Torsion}$ is free, thus $H^n(X)_{\mathrm{ph}} = \operatorname{Ext}^1(H_{n-1}(X)/\operatorname{Torsion}, \mathbb{Z})$ vanishes. □

Recall that the group of isomorphism classes of extensions of an arbitrary abelian group by a torsion free abelian group is divisible (see [17], page 223, point (I)). Whence Theorem 4.1 implies the following:

Corollary 4.3. *Let X be a CW-complex. Then the groups $H^n(X)_{\mathrm{ph}}$, $n \geq 0$ are divisible groups.*

Recall (see e.g. [17], Theorem 23.1), that any divisible group D is isomorphic to a direct sum of groups of the form $\mathbb{Q}$ and $\mathbb{Z}[p^{-1}]/\mathbb{Z}$, where $p > 0$ runs over all prime numbers. Up to isomorphism, the divisible abelian group D is determined by the invariants

$$\nu_0(D) = \dim_{\mathbb{Q}}(D/\mathrm{Torsion}) \quad \text{and} \quad \nu_p(D) = \dim_{\mathbb{F}_p}(\operatorname{Hom}(\mathbb{F}_p, D)),$$

which should be regarded as cardinal numbers.

Proposition 4.4. *Let $n \geq 2$, and D be a divisible abelian torsion group. Then there is a connected CW-complex X of dimension n so that $H^n(X)_{\mathrm{ph}} = H^n(X)$, and that the torsion part of this group is isomorphic to D. Furthermore, we can choose X with $H^m(X) = 0$ for $m \neq 0, n$.*

Proof. Let κ be the smallest infinite cardinal number $\geq 2^{\mathrm{Card}(I)}$. It follows from [27], proof of Theorem 3, that there is a torsion free group A of cardinality κ so that the divisible groups $\operatorname{Ext}^1(A, \mathbb{Z})$ and D have the same invariant ν_p for all primes $p > 0$, whence the torsion part of $\operatorname{Ext}^1(A, \mathbb{Z})$ is isomorphic to D. Furthermore, it follows from the construction in loc. cit. that $\operatorname{Hom}(A, \mathbb{Z}) = 0$. Clearly, we have a presentation

$$0 \longrightarrow R \longrightarrow F \longrightarrow A \longrightarrow 0,$$

where F is a free group of rank κ, such that R is a free group of rank $\kappa' \leq \kappa$. Let $X = M(A, n-1)$ be the corresponding *Moore space*, whose sole nonvanishing homology group in nonzero degrees is $H_{n-1}(X) = A$. Note that this can be constructed as a CW-complex with only one 0-cell, κ cells of dimension $n-1$, and κ' cells of dimension n. The Universal Coefficient Theorem gives $H^n(X) = \operatorname{Ext}^1(A, \mathbb{Z})$ and $H^m(X) = 0$ for $m \neq n, 0$. According to Theorem 4.1, we have $H^n(X)_{\mathrm{ph}} = \operatorname{Ext}^1(A, \mathbb{Z})$, whence the result. □

Remark 4.5. It appears difficult to make useful statements about the rank ν_0 of $H^n(X)_{\mathrm{ph}}$; the answer seems to depend strongly on the chosen axioms of set theory, compare [26].

Given a CW-complex X, we define the *cohomological phantom Brauer group*

$$\operatorname{Br}'(X)_{\mathrm{ph}} = \operatorname{Br}'(X) \cap H^3(X)_{\mathrm{ph}},$$

the intersection taking place in $H^3(X)$. The *phantom Brauer group* is defined as $\operatorname{Br}(X)_{\mathrm{ph}} = \operatorname{Br}(X) \cap H^3(X)_{\mathrm{ph}}$. One might guess that cohomological phantom

Brauer classes rarely come from projective bundles. Note however that we have the following:

Proposition 4.6. *Let D be a divisible abelian torsion group. Then there is a 3-dimensional CW-complex X so that $\mathrm{Br}(X)_{\mathrm{ph}} \simeq D$, and that the inclusion $\mathrm{Br}(X)_{\mathrm{ph}} \subset \mathrm{Br}'(X)$ is an equality.*

Proof. According to Proposition 4.4, there is a 3-dimensional CW-complex X such that $H^3(X)_{\mathrm{ph}} = H^3(X)$, and that this abelian group has torsion part isomorphic to D. In other words, $\mathrm{Br}'(X) \simeq d$. By [52], we have $\mathrm{Br}(X) = \mathrm{Br}'(X)$, because X is of dimension ≤ 4. □

Phantom cohomology is a special case of phantom maps $f : X \to Y$. These are continuous maps that become homotopic to a constant map on each finite subcomplex $X_\alpha \subset X$, without being homotopic to a constant map themselves. (A variant is to demand that it is homotopic to a constant map on each n-skeleton X^n.) The first published example of phantom maps is due to Adams and Walker [1], who used the space $X = S^1 \wedge \mathbb{CP}^\infty$. Further constructions appear in [20] with $f : \mathbb{CP}^\infty \to S^3$, and in [2] with $f : K(\mathbb{Z}, 2m-1) \to BU$ for $m \geq 2$.

A rather explicit example of phantom cohomology is given in [42], Chapter III, page 136: Let $p > 0$ be a prime number (in loc. cit. the prime $p = 3$ was used), and X be the $\mathbb{Z}[1/p]$-localized n-sphere. Our result allows to compute the phantom cohomology group as a divisible abelian group:

Proposition 4.7. *Let X be the $\mathbb{Z}[1/p]$-localized n-sphere, as above. Then we have*

$$H^{n+1}(X)_{\mathrm{ph}} = H^{n+1}(X) \simeq \mathbb{Z}_p/\mathbb{Z},$$

and this divisible group has invariants $\nu_0 = 2^{\aleph_0}$ and $\nu_p = 0$, whereas $\nu_l = 1$ for all other primes $l \neq p$.

Proof. As explained in loc. cit., the space X is a CW-complex with one 0-cell and countably many n-cells and $(n+1)$-cells. The corresponding boundary map in the cellular chain complex can be written as the matrix

$$A = \begin{pmatrix} 1 & & & \\ -p & 1 & & \\ & -p & 1 & \\ & & \ddots & \ddots \end{pmatrix}.$$

This linear map is injective, thus $H_{n+1}(X) = 0$. One easily computes that the cokernel $H_n(X)$ is isomorphic to $\mathbb{Z}[p^{-1}]$, which also follows from the universal property of localization for CW-complexes. This already gives $H^{n+1}(X)_{\mathrm{ph}} = \mathrm{Ext}^1(H_n(X), \mathbb{Z}) = H^{n+1}(X)$, the first equation by Theorem 4.1, the second by the Universal Coefficient Theorem.

To compute this group, apply the Hom functor to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}[p^{-1}] \to \mathbb{Z}[p^{-1}]/\mathbb{Z} \to 0$, which gives the exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \longrightarrow \mathrm{Ext}^1(\mathbb{Z}[p^{-1}]/\mathbb{Z}, \mathbb{Z}) \longrightarrow \mathrm{Ext}^1(\mathbb{Z}[p^{-1}], \mathbb{Z}) \longrightarrow 0.$$

The term in the middle is isomorphic to the ring of p-adic numbers $\mathbb{Z}_p$ (compare [50], page 74), and it follows that the term on the right is isomorphic to $\mathbb{Z}_p/\mathbb{Z}$.

It remains to determine the invariants for the divisible group $D = \mathbb{Z}_p/\mathbb{Z}$. The Snake Lemma applied to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "l"'] & \mathbb{Z}_p \arrow[r] \arrow[d, "l"'] & D \arrow[r] \arrow[d, "l"'] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}_p \arrow[r] & D \arrow[r] & 0,
\end{tikzcd}$$

where the vertical arrows are multiplication by some prime $l > 0$, immediately reveals that the multiplication map $l : D \to D$ is bijective for $l = p$, whereas it has kernel isomorphic to $\mathbb{Z}/l\mathbb{Z}$ else. This gives the values for ν_l. Using that $\mathbb{Z}_p$ and whence D has cardinality $2^{\aleph_0}$, we infer the value $\nu_0 = 2^{\aleph_0}$. $\square$

5. Plus construction and classifying spaces

By now we have gained a fairly good understanding of Brauer groups for Eilenberg–MacLane spaces of type $K(G, n)$ with $n \geq 2$. In this final section we study the case $n = 1$, that is, classifying spaces for discrete groups. Somewhat surprisingly, this turns out to be, in some sense, the universal situation.

Let X be a CW-complex, and $N \subset \pi_1(X)$ be normal subgroup that is *perfect*, i.e., N coincides with its commutator subgroup $[N, N]$. The *plus construction* $X \subset X^+$ is a relative CW-complex obtained by attaching certain cells of dimension two and three, such that the following holds:

(i) $\pi_1(X) \to \pi_1(X^+)$ is surjective, with kernel N.
(ii) $H_*(X) \to H_*(X^+)$ is bijective.

It has the following universal property: For every CW-complex Y and every continuous map $f : X \to Y$ such that N is contained in the kernel of $\pi_1(X) \to \pi_1(Y)$, there is a unique homotopy class of continuous maps $f^+ : X^+ \to Y$ so that $f^+|_X$ is homotopic to f. The plus construction was first introduced by Kervaire [34], and used by Quillen to define the higher K-groups for rings [41].

Proposition 5.1. *Let X be a CW-complex. Then the restriction maps* $\mathrm{Br}'(X^+) \to \mathrm{Br}'(X)$ *and* $\mathrm{Br}(X^+) \to \mathrm{Br}(X)$ *are bijective.*

Proof. Since $H_2(X) \to H_2(X^+)$ is bijective, the induced map on cohomological Brauer groups is bijective, according to Proposition 1.2. In turn, the map on Brauer groups is injective, and it remains to check that every bundle on X extends to X^+. Since the classifying space BPGL_n is simply connected, every continuous mapping $f : X \to \mathrm{BPGL}_n$ extends to X^+ by the universal property of the plus construction. $\square$

We thus have canonical identifications

$$\mathrm{Br}'(X^+) = \mathrm{Br}'(X) \quad \text{and} \quad \mathrm{Br}(X^+) = \mathrm{Br}(X).$$

Now let Y be a connected CW complex. According to a result of Kan and Thurston [33], there is a group G, a perfect normal subgroup $N \subset G$, and a homotopy equivalence

$$(BG)^+ \longrightarrow Y.$$

Here BG denotes the classifying space of the discrete group G, defined in a *functorial way* as a geometric realization of the nerve of category with a single object and

morphism set G, and $(BG)^+$ is the plus construction with respect to $N \subset G = \pi_1(BG)$. Note that this group G is often uncountable, but in any case we have

$$\mathrm{Br}'(Y) = \mathrm{Br}(BG) \quad \text{and} \quad \mathrm{Br}(Y) = \mathrm{Br}(BG)$$

by Proposition 5.1. Concerning the question of equality of $\mathrm{Br}(X) \subset \mathrm{Br}'(X)$, we are thus reduced to the case $X = BG$ for discrete groups G. This translates part of the question from algebraic topology into group theory, as we have

$$\mathrm{Br}'(BG) = \mathrm{Torsion}\,\mathrm{Ext}^1(H_2(G)/\mathrm{Divisible}, \mathbb{Z})$$

by Proposition 1.2. Recall that $H_2(G) = H_2(G, \mathbb{Z})$ is often referred to as the *Schur multiplier*. For G finite, the Schur multiplier is finite, and thus non-canonically isomorphic to $\mathrm{Br}'(BG)$.

Let us now recall a fundamental notion from abelian group theory: Suppose G is an abelian torsion group. A subgroup $H \subset G$ is called *basic* if the following three conditions hold:

(i) The group H is isomorphic to a direct sum of cyclic groups.
(ii) The residue class group G/H is divisible.
(iii) For each integer n, the inclusion $nH \subset nG \cap H$ is an equality.

Basic subgroups $H \subset G$ always exists, and the isomorphism class of the abstract group H is unique. The theory of basic subgroups goes back to Kulikof [35], who treated the case that G is *p-primary*, that is, the orders of group elements are p-power. For the general situation, compare [17], Chapter VI, Section 33.

Theorem 5.2. *Let $p > 0$ be a prime, and G be a p-primary torsion group whose basic subgroups $H \subset G$ are infinite. Then the classifying space $X = BG$ has the property* $\mathrm{Br}(X) \subsetneqq \mathrm{Br}'(X)$.

This applies in particular to the case that G is an infinite-dimensional $\mathbb{F}_p$-vector space. The proof relies on a series of facts, which we have to establish first. To begin with, recall that for every abelian group, the addition map $G \times G \to G$ induces on the graded homology module $H_*(G)$ the structure of an alternating associative ring. Therefore, the canonical bijection $G \to H_1(G)$ yields a ring homomorphism $\Lambda^*(G) \to H_*(G)$ from the exterior algebra to the homology ring.

Proposition 5.3. *For every abelian group G, the canonical map $\Lambda^2(G) \to H_2(G)$ is bijective. If G is torsion and $H \subset G$ is a basic subgroup, the induced map $H_2(H) \to H_2(G)$ is bijective as well.*

Proof. The first assertion appears in [11], Chapter V, Theorem 6.4. Now let G be torsion, and $H \subset G$ a basic subgroup. The task is to check that the canonical map

$$\Lambda^2(H) \longrightarrow \Lambda^2(G)$$

is bijective. We start with surjectivity. Let $g \wedge g' \in \Lambda^2(G)$. Choose $n > 0$ with $ng' = 0$. Using that G/H is divisible, we may write $g = na + h$ with $a \in G$ and $h \in H$. Then $g \wedge g' = h \wedge g' = -g' \wedge h$. Repeating the argument, we see that $g \wedge g' = h \wedge h'$ for some $h, h' \in H$.

It remains to check injectivity. Suppose we have a nonzero element $x \in \Lambda^2(H)$. Since H is a direct sum of finite cyclic groups, so is $\Lambda^2(H)$. This implies that there is an integer $n \neq 0$ with $x \notin n\Lambda^2(H)$. Since $nH = H \cap nG$, the canonical map of

$\mathbb{Z}/n\mathbb{Z}$-modules $H/nH \to G/nG$ is injective (in fact bijective). It follows that the image of x in

$$\Lambda^2(H) \otimes \mathbb{Z}/n\mathbb{Z} = \Lambda^2(H/nH) = \Lambda^2(G/nG)$$

is nonzero, thus the same holds in $\Lambda^2(G)$. □

Corollary 5.4. *Let G be an abelian torsion group, with basic subgroup H of the form $H = \bigoplus_{i\in I} \mathbb{Z}/n_i\mathbb{Z}$. Then the cohomological Brauer group $\mathrm{Br}'(X)$ of the classifying space $X = BG$ is isomorphic to the torsion part of $\prod_{i<j} \mathbb{Z}/(n_i, n_j)$.*

Proof. We have $H_2(G) = \Lambda^2(H)$ by the Proposition. Using $\Lambda^2(\mathbb{Z}/n_i\mathbb{Z}) = 0$, one obtains a decomposition

$$\Lambda^2(H) = \bigoplus_{i<j} \Lambda^1(\mathbb{Z}/n_i\mathbb{Z}) \otimes \Lambda^1(\mathbb{Z}/n_j\mathbb{Z}) = \bigoplus_{i<j} \mathbb{Z}/(n_i, n_j).$$

Here we have chosen a total order on the index set I. Proposition 1.2, together with the fact that $\mathrm{Ext}^1(\mathbb{Z}/d\mathbb{Z}, \mathbb{Z})$, $d \neq 0$ is non-canonically isomorphic to $\mathbb{Z}/d\mathbb{Z}$, now yields the result. □

Next, we need a result that allows us to pass back, in rather special circumstances, from bundles over classifying spaces to representations. Given a representation $\rho : \pi_1(X) \to L$ in some Lie group L, we obtain an associated L-bundle $V \to X$, which is defined as the quotient of $L \times X$ modulo the diagonal action of the fundamental group. Such bundles are called *flat*. We are mainly interest in the case $X = BG$, where $\pi_1(X) = G$, and $L = \mathrm{PU}(n) \subset \mathrm{PGL}_n$.

Proposition 5.5. *Let $p > 0$ be a prime, and G be a group that is an ascending union of subgroups $G_0 \subset G_1 \subset \ldots$. Suppose that the G_j are finite p-groups and that the inclusion homomorphisms $G_j \subset G_{j+1}$ admit left inverses. Then every PGL_n-bundle on the classifying space $X = BG$ is associated to a representation $G \to \mathrm{PU}(n)$.*

Proof. We have $BG = \bigcup_j BG_j$. Let $V \to BG$ be a PGL_n-bundle of rank $n \geq 0$. and write $V_j = V|BG_j$ for the restrictions. By a result of Dwyer and Zabrodsky [13], the canonical maps

$$\mathrm{Hom}(G_i, \mathrm{PU}(n)) \longrightarrow [BG_i, \mathrm{BPU}(n)] = [BG_i, \mathrm{BPGL}_n]$$

are surjective. Choose a representation $\rho_j : G_j \to \mathrm{PU}(n)$ so that V_j is isomorphic to the bundle associated to ρ_j. For each j, the representations $\rho_{j+1}|G_j$ and ρ_j are conjugate. This follows from [51], Chapter V, Corollary 4.4. Replacing inductively the ρ_{j+1} by suitable $a_j^{-1}\rho_{j+1}a_j$, we may assume that $\rho_{j+1}|G_j = \rho_j$. In turn, we get a representation $\rho : G \to \mathrm{PU}(n)$ with $\rho|G_j = \rho_j$. Consider the associated PGL_n-bundle $W \to BG$. By construction, the bundles V, W restrict to isomorphic bundles on each BG_j.

It remains to verify that two PGL_n-bundles V, W of rank n that are isomorphic on each BG_j are isomorphic. Let V_j, W_j be the restrictions to BG_j. Since $V = \bigcup V_j$ and $W = \bigcup W_j$, it suffices to construct compatible bundle isomorphism $f_j : V_j \to W_j$. We do this by induction on $j \geq 0$. Choose an arbitrary isomorphism f_0, and suppose we already have a compatible family $f_0, \ldots, f_j$. Let $P_{j+1} \to BG_{j+1}$ be the PGL_n-torsor of bundle isomorphism $V_{j+1} \to W_{j+1}$. Then $P_{j+1}|BG_j$ is isomorphic to $P_j \to BG_j$. By assumption, all these torsors admit a section, thus are isomorphic to the trivial torsor. To conclude the proof, it therefore suffices to verify that every

continuous function $BG_j \to \mathrm{PGL}_n$ extends along the inclusion $BG_j \subset BG_{j+1}$. But this is trivial, because our left inverses $l_{j+1} : G_{j+1} \to G_j$ for the inclusions $G_j \subset G_{j+1}$ induce by functoriality continuous maps $Bl_{j+1} : BG_{j+1} \to BG_j$ that are the identity on BG_j. □

Proposition 5.6. *Let G be an abelian group, and $V \to BG$ the PGL_n-bundle associated to a representation $\rho : G \to \mathrm{PU}(n)$. Then there is a subgroup $M \subset G$ of finite index so that the restriction $V|BM$ comes from a vector bundle of rank n.*

Proof. This is essentially a result on unitary representations of discrete groups due to Backhouse and Bradley [7]. For the sake of the reader, we briefly recall parts of their argument in our setting. For each $g \in G$, choose a matrix $A_g \in \mathrm{SU}(n)$ representing $\rho(g) \in \mathrm{PU}(n)$. Make the choice so the A_e is the unit matrix for the neutral element $e \in G$. Now define the *multiplier* $\omega_{g,h} \in \mu_n$ by the formula

$$A_g \cdot A_h = \omega_{g,h} \cdot A_{gh}.$$

Then the 2-cochain $\omega_{g,h}$ is a cocycle, and its cohomology class in $H^2(G, \mathbb{C}^\times)$ is precisely the obstruction against lifting the projective representation ρ to a linear representation.

On the other hand, one may use this 2-cocycle to endow the set $\tilde{G} = \mu_\infty \times G$ with the structure of a central extension of G, by declaring the group law as

$$(\zeta, g) \cdot (\xi, h) = (\zeta\xi\omega_{g,h}, gh).$$

Here $\mu_\infty \subset \mathbb{C}^\times$ is the group of all complex roots of unity. The representation $\rho : G \to \mathrm{PU}(n)$ yields a unitary representation $\tilde{\rho} : \tilde{G} \to U(n)$, defined via $\tilde{\rho}(\zeta, g) = \zeta A_g$. We may assume that the latter linear representation is irreducible, by passing to an irreducible subrepresentation. This replaces n by some $n' \leq n$, but does not affect the multiplier.

Using Zorn's Lemma, one checks that there is a maximal subgroup $M \subset G$ on which the multiplier is symmetric, that is, $\omega_{g,h} = \omega_{h,g}$ for all $g, h \in M$. Consider the induced central extension $\tilde{M} = \tilde{G} \times_G M$ of M. Obviously, the group $\tilde{M}$ is abelian, and we may view $0 \to \mu_\infty \to \tilde{M} \to M \to 0$ as an extension in the category of abelian groups. The latter extension splits, because μ_∞ is divisible and hence $\mathrm{Ext}^1(M, \mu_\infty)$ vanishes. Consequently, the representation $\rho : M \to \mathrm{PU}(n) \subset \mathrm{PGL}_n$ lifts to a linear representation $M \to \mathrm{GL}_n$.

It remains to verify that the subgroup $M \subset G$ has finite index. This indeed holds by [7], Theorem 3. Note that in loc. cit. the irreducibility of the unitary representation of $\tilde{G}$ enters. There it is also shown that all irreducible unitary projective representation with multiplier ω have the same dimension $d(\omega)$, and that this dimension coincides with the index of $M \subset G$. □

Proof of Theorem 5.2: Let $p > 0$ be a prime and G be a p-primary torsion group whose basic subgroups $H \subset G$ are infinite. Write $H = \bigoplus_{i \in I} \mathbb{Z}/p^{\nu_i}\mathbb{Z}$. By the proof for Corollary 5.4, the cohomological Brauer group of BG is the torsion part of

$$\mathrm{Ext}^1(\Lambda^2(H), \mathbb{Z}) = \mathrm{Hom}(\Lambda^2(H), \mathbb{Q}/\mathbb{Z}) = \prod_{i<j} \mathrm{Hom}(\mathbb{Z}/(p^{\nu_i}, p^{\nu_j}), \mathbb{Q}/\mathbb{Z}). \tag{2}$$

In particular, the canonical map $BH \to BG$ induces a bijection on cohomological Brauer groups. Replacing G by H, we may assume that G itself is a direct sum of cyclic groups. Restricting to a direct summand and permuting the summands, we

may assume that the index set I is the set of natural numbers, and that $\nu_i \leq \nu_j$ for $i \leq j$.

Set $X = BG$, and let $\alpha \in \mathrm{Br}'(X)$ be a torsion element. Write it as a tuple $\alpha = (\alpha_{ij})_{i<j}$ with respect to the decomposition (2). For each $i \in I$, let $J_i \subset I$ be the set of $j \in I$ with $i < j$ and $\alpha_{ij} \neq 0$. Furthermore, let $I_{\mathrm{fin}} \subset I$ be the set of all $i \in I$ with J_i finite. *Now suppose that α has the property that $I \smallsetminus I_{\mathrm{fin}}$ is finite, and that the cardinalities of the sets J_i, $i \in I_{\mathrm{fin}}$ are unbounded.* We claim that any such α does not lie in the Brauer group.

Suppose to the contrary that there is a PGL_n-bundle $V \to X$ whose obstruction class is α. Clearly, the assumptions of Proposition 5.5 hold for the subgroups $G_j = \bigoplus_{i=0}^{j} \mathbb{Z}/p^{\nu_i}\mathbb{Z}$ of G; hence our projective bundle is associated to a projective representation $\rho : G \to \mathrm{PGL}_n$. By Proposition 5.6, there is a subgroup $M \subset G$ of finite index so that the projective representation lifts to a linear representation of the subgroup M. Thus $\alpha|\Lambda^2(M) = 0$.

To reach a contradiction, it remains to check that α remains nonzero on $\Lambda^2(M)$ for each subgroup $M \subset G$ of finite index. We do this by induction on the index $[G : M]$. The case $M = G$ is trivial, so let us assume $M \subsetneqq G$. Choose a short exact sequence

$$0 \longrightarrow G' \longrightarrow G \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0. \tag{3}$$

with $M \subset G'$. We shall see that for some other subgroup $U \subset G'$, we may apply the induction hypothesis to $M \cap U \subset U$. Indeed, we have

$$U/(M \cap U) = (U + M)/M \subsetneqq G/M,$$

so $[U : M \cap U] < [G : M]$, and the problem is to choose U admitting a suitable direct sum decomposition, so that $\alpha|\Lambda^2(U)$ retains its properties with respect to the new decomposition.

With respect to the given direct sum decomposition $G = \bigoplus_{i \geq 0} \mathbb{Z}/p^{\nu_i}\mathbb{Z}$, the map on the right in (3) can be viewed as a matrix $(\lambda_0, \lambda_1, \ldots)$ with $\lambda_k \in \mathbb{Z}/p\mathbb{Z}$. First, consider the case that $\lambda_k = 0$ for almost all $k \in I$, say for all $k \geq k_0$. Then obviously $U = \bigoplus_{i \geq k_0} \mathbb{Z}/p^{\nu_i}\mathbb{Z}$ does the job.

Suppose now that $\lambda_k \neq 0$ for infinitely many $k \in I$. Then there must be such an index with J_k finite. Without restriction, we may assume $\lambda_k = 1$. For $i \neq k$, choose lifts $\lambda'_i \in \mathbb{Z}$ of $\lambda_i \in \mathbb{Z}/p\mathbb{Z}$, and consider $e'_i = e_i - \lambda'_i e_k \in G$. Also set $e'_k = e_k$. Then the elements $e'_i \in G$, $i \geq 0$ form a new "basis", and the e'_i, $i \neq k$ generate a subgroup $U \subset G$ contained in G'. In $\Lambda^2(G)$, we obviously have

$$e'_i \wedge e'_j = e_i \wedge e_j - \lambda'_i e_k \wedge e_j + \lambda'_j e_k \wedge e_i.$$

Decomposing $\alpha = (\alpha'_{ij})_{i<j}$ with respect to the new "basis" $e'_i \in G$ as in (2), the above formula shows that for all $i > k$, $i \notin J_k$, the condition $\alpha_{ij} \neq 0$ is equivalent to $\alpha'_{ij} \neq 0$, except for at most $\mathrm{Card}(J_k)$ indices $j \in J_k$. It follows that the new tuple $(\alpha'_{ij})_{i<j}$ has the same properties as the original tuple $(\alpha_{ij})_{i<j}$, and the same holds if we restrict to indices $i, j \neq k$. The induction hypothesis applied to $M \cap U \subset U$ tells us that α remains nonzero on $\Lambda^2(M \cap U)$, thus in particular on $\Lambda^2(M)$. □

References

[1] J. Adams, G. Walker: An example in homotopy theory. Math. Proc. Cambr. Philos. Soc. (1964), 699–700.

[2] D. Anderson, L. Hodkin: The K-theory of Eilenberg-MacLane complexes. Topology 7 (1968), 317–329.
[3] B. Antieau, B. Williams: The period-index problem for twisted topological K-theory Preprint, arXiv:1104.4654.
[4] B. Antieau, B. Williams: On the classification of principal PU_2-bundles over a 6-complex. Preprint, arXiv:1209.2219.
[5] S. Araki, Z.-I. Yosimura: A spectral sequence associated with a cohomology theory of infinite CW-complexes. Osaka J. Math. 9 (1972), 351–365.
[6] M. Atiyah, G. Segal: Twisted K-theory. Ukr. Math. Bull. 1 (2004), 291–334.
[7] N. Backhouse, C. Bradley: Projective representations of abelian groups. Trans. Amer. Math. Soc. 16 (1972), 260–266.
[8] C.-F. Bödigheimer: Remark on the realization of cohomology groups. Quart. J. Math. Oxford 34 (1983), 1–5.
[9] G. Bredon: Sheaf theory. McGraw-Hill Book Co., New York-Toronto-London, 1967.
[10] G. Bredon: A space for which $H^1(X;Z) \not\approx [X,S^1]$. Proc. Amer. Math. Soc. 19 (1968), 396–398.
[11] K. Brown: Cohomology of groups. Springer, Berlin, 1982.
[12] A. de Jong: A result of Gabber. Preprint, http://www.math.columbia.edu/∼dejong/
[13] W. Dwyer, A. Zabrodsky: Maps between classifying spaces. In: J. Aguadé and R. Kane (eds.), Algebraic topology, pp. 106–119. Lect. Notes Math. 1298. Springer, Berlin, 1987.
[14] D. Edidin, B. Hassett, A. Kresch, A. Vistoli: Brauer groups and quotient stacks. Amer. J. Math. 123 (2001), 761–777.
[15] P. Eklof, A. Mekler: Almost free modules. Set-theoretic methods. North-Holland, Amsterdam, 2002.
[16] R. Fritsch, R. Piccinini: Cellular structures in topology. Cambridge University Press, Cambridge, 1990.
[17] L. Fuchs: Infinite abelian groups. I. Academic Press, New York-London, 1970.
[18] J. Giraud: Cohomologie non abélienne. Springer, Berlin, 1971.
[19] R. Godement: Topologie algébrique et théorie des faisceaux. Hermann, Paris, 1964.
[20] B. Gray: Spaces of the same n-type, for all n. Topology 5 (1966), 241–243.
[21] A. Grothendieck: A general theory of fibre spaces with structure sheaf. University of Kansas, Department of Mathematics, Report No. 4.
[22] A. Grothendieck: Sur quelques points d'algèbre homologique. Tohoku Math. J. 9 (1957), 119–221.
[23] A. Grothendieck: Le groupe de Brauer. I. Algèbres d'Azumaya et interprétations diverses. In: J. Giraud (ed.) et al.: Dix exposés sur la cohomologie des schémas, pp. 46–189. North-Holland, Amsterdam, 1968.
[24] M. Hakim: Topos anneles et schemas rélatifs. Springer, Berlin-Heidelberg-New York, 1972.
[25] H. Hamm: Zum Homotopietyp Steinscher Räume. J. Reine Angew. Math. 338 (1983), 121–135.
[26] H. Hiller, M. Huber, S. Shelah: The structure of Ext(A,Z) and V=L. Math. Z. 162 (1978), 39–50.
[27] M. Huber, R. Warfield: On the torsion subgroup of Ext(A,G). Arch. Math. 32 (1979), 5–9.
[28] P. Huber: Homotopical cohomology and Čech cohomology. Math. Ann. 144 (1961), 73–76.
[29] D. Husemoller: Fibre bundles. Berlin, Springer, 1993.
[30] K. Ishiguro: Unstable Adams operations on classifying spaces. Math. Proc. Cambridge Philos. Soc. 102 (1987), 71–75.
[31] S. Jackowski, J. McClure, B. Oliver: Homotopy classification of self-maps of BG via G-actions. I. Ann. of Math. 135 (1992), 183–226.
[32] C. Jensen: Les foncteurs dérivés de $\varprojlim$ et leurs applications en théorie des modules. Lect. Notes Math. 254. Springer, Berlin-New York, 1972.
[33] D. Kan, W. Thurston: Every connected space has the homology of a $K(G,1)$. Topology 15 (1976), 253–258.
[34] M. Kervaire: Smooth homology spheres and their fundamental groups. Trans. Amer. Math. Soc. 144 (1969) 67–72.
[35] L. Kulikov: On the theory of abelian groups of arbitrary cardinality. Mat. Sb. 16 (1945), 129–162.

[36] C. McGibbon: Phantom maps. In: I. James (ed), Handbook of algebraic topology, pp. 1209–1257. North-Holland, Amsterdam, 1995.
[37] A. Mekler, S. Shelah: Every coseparable group may be free. Israel J. Math. 81 (1993), 161–178.
[38] J. Milnor: On axiomatic homology theory. Pacific J. Math. 12 (1962), 337–341.
[39] M. Mimura: Homotopy theory of Lie groups. In: I. James (ed.), Handbook of algebraic topology, pp. 951–991. North-Holland, Amsterdam, 1995.
[40] D. Notbohm: Maps between classifying spaces and applications. J. Pure Appl. Algebra 89 (1993), 273–294.
[41] D. Quillen: Cohomology of groups. Actes du Congrès International des Mathématiciens, Tome 2, pp. 47–51. Gauthier-Villars, Paris, 1971.
[42] Y. Rudyak: On Thom spectra, orientability, and cobordism. Springer, Berlin, 1998.
[43] C. Schochet: A Pext primer: pure extensions and $\lim^1$ for infinite abelian groups. NYJM Monographs, Albany, NY, 2003.
[44] S. Schröer: There are enough Azumaya algebras on surfaces. Math. Ann. 321 (2001), 439–454.
[45] S. Schröer: Topological methods for complex-analytic Brauer groups. Topology 44 (2005), 875–894.
[46] S. Schröer: Pathologies in cohomology of non-paracompact Hausdorff spaces. Preprint, http://reh.math.uni-duesseldorf.de/∼schroeer/publications.html
[47] J.-P. Serre: Groupes d'homotopie et classes de groupes abliens. Ann. of Math. 58 (1953). 258–294.
[48] J.-P. Serre: Cohomologie galoisienne. Fifth edition. Lect. Notes Math. 5. Springer, Berlin, 1994.
[49] D. Sullivan: Geometric topology: localization, periodicity and Galois symmetry. Springer, Dordrecht, 2005.
[50] C. Weibel: An introduction to homological algebra. Cambridge University Press, Cambridge, 1994.
[51] G. Whitehead: Elements of homotopy theory. Springer-Verlag, New York-Berlin, 1978.
[52] L. Woodward: The classification of principal PU_n-bundles over a 4-complex. J. London Math. Soc. 25 (1982), 513–524.
[53] J. Wiegold: Ext(Q,Z) is the additive group of real numbers. Bull. Austral. Math. Soc. 1 (1969), 341–343.

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